Journal of Statistical Physics, Vol. 75, Nos. 1/2, 1994

Diffusion on the Torus for Hamiltonian Maps

S. Siboni,[1,2] G. Turchetti,[1] and S. Vaienti[2,3]

Received April 28, 1993; final January 4, 1994

For a mapping of the torus $\mathbb{T}^2$ we propose a definition of the diffusion coefficient D suggested by the solution of the diffusion equation on $\mathbb{T}^2$. The definition of D, based on the limit of moments of the invariant measure, depends on the set Ω where an initial uniform distribution is assigned. For the algebraic automorphism of the torus the limit is proved to exist and to have the same value for almost all initial sets Ω in the subfamily of parallelograms. Numerical results show that it has the same value for arbitrary polygons Ω and for arbitrary moments.

KEY WORDS: Decay of correlation; diffusion process.

1. INTRODUCTION

The transport in chaotic regions of phase space of Hamiltonian maps or flows is quite relevant for many physiucal systems[1] (a confined plasma, the beam of a particle accelerator, a spinning planet or a galaxy). A theory of transport is still missing due to its extreme complexity even for systems with a small number of degrees of freedom.[2-8] Indeed if we consider an integrable map of the plane sufficiently perturbed, then a variety of new topological structures appear, such as Cantoris issuing from the breakup of KAM curves, chains of islands, chaotic regions issuing from homoclinic and etheroclinic intersections of hyperbolic manifolds, with replicas continuing ad infinitum under scale changes.

The picture simplifies somewhat when the perturbation strength is increased since the measure of the "chaotic regions" increases, but unlike the limit of vanishing perturbation where the measure of the KAM curves

---

[1] Istituto di Fisica dell'Università Bologna and INFN Sezione Bologna, Bologna, Italy.
[2] Centre de Physique Théorique, CNRS Luminy, Marseille, France.
[3] Département de Mathématique Université de Toulon et du Var, France.

0022-4715/94/0400-0167$07.00/0

equals the measure of all orbits, one cannot show that the measure of surviving islands converges uniformly and rapidly to zero.

The description should be made on a local basis with "transport coefficients" varying in phase space, to take into account the coexistence of regions of stochastic and regular motion, where no diffusion can occur.[9] The symmetries of the unperturbed system determine the choice of the coordinates at least for moderate perturbations.

If an area-preserving map is described by action–angle coordinates, $j \in \mathbb{R}$, $\theta \in \mathbb{T}$, it is usually believed that the angle, being a fast variable, randomizes on a time interval small with respect to the time scale on which the action shows an appreciable diffusion.[10,11] As a consequence it is quite reasonable to assume a Markov property for the action j and to compare the asymptotic distribution of j with the solution $\rho(j; t)$, of a Fokker–Planck (FP) equation for an initial distribution $\rho(j; 0)$. The diffusion coefficient $D(j)$ has to be determined from the knowledge of $\rho(j; t)$ by solving an inverse problem. A simple solution is found when $D(j) = D$ is constant. Given a uniform initial distribution on a set $\Omega \subset \mathbb{R} \times \mathbb{T}$, then $\rho(j; 0)$ depends on Ω and so does $\rho(j; t)$, given by the convolution of $\rho(j; 0)$ with a Gaussian. The simplest way to recover D is to compute the even moments, recalling that

$$D = \left\{ \frac{q!}{(2q)!} \lim_{t \to \infty} \frac{1}{t^q} \langle [j(t) - j(0)]^{2q} \rangle_\Omega \right\}^{1/q} \tag{1.1}$$

The equality holds for any t for a FP equation with constant D; the limit is necessary if we assume that the FP equation is only asymptotically (in t) satisfied.

Usually the diffusion coefficient is defined by the second moment for a specific domain Ω. However, its existence is not sufficient to determine the statistical properties of the process; to this end the higher moments should be considered and their independence from the set Ω verified.

We notice also that the solution of the Fokker–Planck equation depends on the manifold on which the process occurs and accordingly change the relations between the asymptotic behavior of the second or higher moments and the diffusion coefficient. For the case of the $\mathbb{T}^2$ torus the diffusion coefficient is related to the moments of order q by

$$D = \lim_{t \to +\infty} -\frac{1}{t} \log |\langle j^q(t) \rangle_\Omega - \langle j^q(\infty) \rangle_\Omega| \left(\frac{\mu_L(\mathbb{T})}{2\pi} \right)^2 \tag{1.2}$$

where μ_L denotes the Lebesgue measure.

We consider here a map definited on the $\mathbb{T}^2$ torus and propose the above definition of the diffusion coefficient, suggested by the asymptotic behavior of the solution of the Fokker–Planck equation. Even in this case, if the limit exists it should be the same for all moments and almost any initial domain Ω.

The main result of this paper is that for the total automorphism the limit exists for the second moment $q = 2$ and for a wide choice of sets $\Omega \subset \mathbb{T}^2$ has the same value $D = 2 \log \lambda \ [\mu_L(\mathbb{T})/2\pi]^2$, where $\log \lambda$ is the positive Lyapunov exponent. More precisely the limit is proved for almost all parallelograms of $\mathbb{T}^2$. Numerical simulations show the result is still the same if other moments $q \neq 2$ are considered and Ω is a polygon. In this respect it is more general than the result[10] for the same map of the cylinder, since in that case the existence of the limit is proved only when $\Omega = \mathbb{T}^2$, the invariant set of the map.

A further step would be to analyze the distribution in both the angle and the action by considering the diffusion matrix, knowledge of which allows one to obtain the diffusion coefficient for any other dynamical variable by a simple change of coordinates in the FP equation.

To conclude we observe that the analysis of the diffusive behavior on compact sets is physically justified. Indeed, area-preserving maps create invariant regions, such as the annulus between two invariant curves, where the dynamics can be chaotic and an analysis of diffusion is justified. In this case the approach to equilibrium is a relaxation process, whereas for noncompact sets, such as the cylinder, a true diffusion takes place. With some abuse of language we use the word diffusion in both cases throughout this paper.

2. DIFFUSION COEFFICIENTS

We consider first an area-preserving map of the cylinder $\mathbb{R} \times \mathbb{T}$, where $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, so that $\mu_L(\mathbb{T}) = 1$. The diffusion coefficient is defined by

$$D_2(\Omega) = \lim_{n \to +\infty} \frac{1}{2n} \mathbf{E}_\Omega((j_n - j_0)^2) \tag{2.1}$$

where $j_n \in \mathbb{R}$ is the action j iterated n times, and $\mathbf{E}_\Omega$ denotes the expectation value computed with respect to initial data, uniformly distributed in a domain $\Omega \subset \mathbb{R} \times \mathbb{T}$. (Having normalized the angle θ to $[0, 1]$, the action is defined by $j = \oint p \, dq$). Such a definition is motivated by an underlying assumption that the angle randomizes rapidly so that a diffusion equation asymptotically holds

$$\frac{\partial \rho}{\partial t} = D \frac{\partial^2 \rho}{\partial j^2} \tag{2.2}$$

where D is a constant coefficient. Given a uniform initial distribution on Ω, then the initial distribution of the action, which we denote j_0, is given by

$$\rho(j_0;0)=\frac{1}{\mu_L(\Omega)}\int_{\mathbb{T}} d\theta_0\, \mathscr{X}_\Omega(j_0,\theta_0) \tag{2.3}$$

where μ_L is the Lebesgue measure and $\mathscr{X}_\Omega$ is the characteristic function of the set Ω. $\rho(j_0;0)$ is itself uniform only if Ω is a direct product such as $\Omega=[a,b]\times\mathbb{T}$. Assuming (2.2) holds for $t\geqslant 0$, the distribution of j at time t reads

$$\rho(j;t)=\int_{\mathbb{R}} dj_0\,\frac{\exp[-(j-j_0)^2/4Dt]}{(4\pi Dt)^{1/2}}\,\rho(j_0;0) \tag{2.4}$$

and for the second moment one has

$$\langle j^2(t)\rangle_\Omega=\int_{\mathbb{R}} dj\, j^2\rho(j;t)=2Dt+\langle j^2(0)\rangle_\Omega \tag{2.5}$$

where $\langle j^2(0)\rangle_\Omega=\int_{\mathbb{R}} dj_0\, j_0^2\rho(j_0;0)$ is the second moment of the initial distribution. As a consequence we can write

$$D=\lim_{t\to+\infty}\frac{1}{2t}\langle[j(t)-j(0)]^2\rangle_\Omega \tag{2.6}$$

If (2.2) holds for $t\geqslant 0$, the limit is unnecessary in (2.6); its presence is required if (2.2) is only asymptotically satisfied.

The existence of the limit (1.1) even for the simplest models like the standard map

$$M:\quad \begin{cases} j_n=j_{n-1}+g(\theta_{n-1}) \\ \theta_n=\theta_{n-1}+j_n \quad \mathrm{mod}\ 1 \end{cases} \tag{2.7}$$

where $g(\theta)$ is a periodic function of $\mathbb{T}$ of zero mean, is not yet known. In this case if we choose $\Omega=\mathbb{T}^2$ the r.h.s. of (2.1) assumes a simpler form since $\mathbb{T}^2$ is an invariantg set of the map. As a consequence one has

$$\begin{aligned} D_2(\mathbb{T}^2)&=\lim_{n\to+\infty}\frac{1}{2n}\mathbf{E}_{\mathbb{T}^2}\left[\left(\sum_{j=0}^{n-1} g(\theta_j)\right)^2\right] \\ &=\frac{1}{2}\mathbf{E}_{\mathbb{T}^2}(g^2(\theta))+\sum_{l=1}^{\infty}\mathbf{E}_{\mathbb{T}^2}[g(\theta)\,g(\theta_l)] \end{aligned} \tag{2.8}$$

where $\theta_l=[M^l(j_0,\theta_0)]_\theta$, $\forall l\in\mathbb{N}$, and the existence of the diffusion coefficient is ensured by the convergence of the series, namy by a suitable decay of the correlation functions $\mathbf{E}_\Omega[g(\theta)\,g(\theta_l)]$.

A necessary condition for the existence of a limit process described by a FP equation is that $D_2(\Omega)$ does not depend on Ω. Moreover, the coefficients

$$D_{2q}(\Omega)=\left\{\frac{q!}{(2q)!}\lim_{n\to\infty}\frac{1}{n^q}\,\mathbf{E}_\Omega[(j_n-j_0)^{2q}]\right\}^{1/q} \tag{2.9}$$

corresponding to the even moments should be independent of q, while the coefficients $D_{2q+1}(\Omega\pi)$ related to the odd moments should vanish.

If the domain on which our map is defined is compact like a 2-torus $\mathbb{T}^2$, the definition of the diffusion coefficient must be changed. Indeed the solution of the diffusion equation (2.2), letting $\mathbb{T}=\mathbb{R}\backslash\mathbb{Z}$, now reads

$$\rho(j;t)=\int_{\mathbb{T}} dj_0\left[1+2\sum_{k=1}^{\infty} e^{-D(2\pi k)^2 t}\cos 2\pi k(j-j_0)\right]\rho(j_0;0) \tag{2.10}$$

The coefficient D is related to the moments of the distribution

$$D=\lim_{t\to+\infty}-\frac{(2\pi)^{-2}}{t}\log|\langle j^q(t)\rangle_\Omega-\langle j^q(\infty)\rangle_\Omega| \tag{2.11}$$

for almost any Ω. This suggests a definition of the diffusion coefficient of the map according to

$$D_q(\Omega)=\lim_{n\to+\infty}-\frac{(2\pi)^{-2}}{n}\log|\mathbf{E}_\Omega(j_n^q)-\mathbf{E}_\Omega(j_\infty^q)| \tag{2.12}$$

In order to take the angle relaxation into account, we consider the FP equation for the distribution function in both variables θ, j. Assuming that both diffusion coefficients are constant, the FP equation reads

$$\frac{\partial\rho}{\partial t}=D\frac{\partial^2\rho}{\partial j^2}+\bar{D}\frac{\partial^2\rho}{\partial\theta^2} \tag{2.13}$$

defined on the torus $\mathbb{T}^2$. The solution corresponding to an initial uniform distribution in a set $\Omega\subset\mathbb{T}^2$

$$\rho(j_0,\theta_0;0)=\frac{\mathcal{X}_\Omega(j_0,\theta_0)}{\mu_L(\Omega)} \tag{2.14}$$

is given by

$$\rho(j,\theta;t)=\int_{\mathbb{T}} dj_0\, G(j\,|\,j_0;t)\int_{\mathbb{T}} d\theta_0\,\bar{G}(\theta\,|\,\theta_0;t)\,\rho(j_0,\theta_0;0) \tag{2.15}$$

where due to separability $\bar{G}$ is the Green's function of the one-dimensional FP equation defined on the one-dimensional torus

$$\bar{G}(\theta \mid \theta_0; t) = 1 + 2 \sum_{k=1}^{\infty} \exp[-\bar{D}(2\pi k)^2 t] \cos 2\pi k(\theta - \theta_0) \tag{2.16}$$

and $G(j \mid j_0; t)$ is given by (2.16) changing θ, θ_0, $\bar{D}$ into j, j_0, D.

If the equation is defined on the cylinder, then the first integral in (2.15) is taken on $\mathbb{R}$ and G is given by

$$G(j \mid j_0; t) = \frac{1}{(4\pi Dt)^{1/2}} \exp\left[-\frac{(j - j_0)^2}{4Dt}\right] \tag{2.17}$$

Assuming $\bar{D} \gg D$, we see that for $t \gg \bar{D}^{-1}$ we have $\bar{G} \sim 1$ and therefore the distribution function (2.15) is identical to (2.10) and satisfies the one-dimensional equation (2.2). The integrated density $\rho(j; t)$

$$\rho(j; t) = \int_{\mathbb{T}} d\theta \, \rho(j, \theta; t) \tag{2.18}$$

satisfies (2.2) and agrees with (2.10) if the phase space is $\mathbb{T}^2$ and with (2.4) if it is $\mathbb{R} \times \mathbb{T}$.

Referring from now on only to the diffusion equation on the torus $\mathbb{T}^2$, we introduce the measure μ whose density is $\rho(j, \theta; t)$ given by (2.15) and corresponds to an initial uniform density on Ω. The measure of a set B, denoted $\mu(B \mid \Omega; t)$, is given by

$$\mu(B \mid \Omega; t) = \frac{1}{\mu_L(\Omega)} \int_{\mathbb{T}^2} dj \, d\theta \, \mathscr{X}_B(j, \theta)$$
$$\times \int_{\mathbb{T}^2} dj_0 \, d\theta_0 \, G(j \mid j_0; t) \, \bar{G}(\theta \mid \theta_0; t) \, \mathscr{X}_\Omega(j_0, \theta_0) \tag{2.19}$$

where $\mu(B \mid \Omega; 0) = \mu_L(B \cap \Omega)/\mu_L(\Omega)$. The integrated density can be written as

$$\rho(j; t) \, dj = \int_{\mathbb{T}} d\mu(j, \theta \mid \Omega; t) = \mu([j, j + dj] \times \mathbb{T} \mid \Omega; t) \tag{2.20}$$

where the integration on $\mathbb{T}$ refers to the angle θ. Its moments read

$$\langle j^q(t) \rangle_\Omega = \int_{\mathbb{T}} dj \, j^q \rho(j; t) = \int_{\mathbb{T}^2} j^q \, d\mu(j, \theta \mid \Omega; t) \tag{2.21}$$

It is evident that for $t \to \infty$ the measure tends to $\mu_L(B)$ and the integrated density to a constant $\rho(j; \infty) = 1$. For almost all choices of B and Ω the difference between $\mu(B \mid \Omega; t)$ and $\mu_L(B)$ is bounded above and below by a constant times $\exp(-4\pi^2 Dt)$. Letting B be a direct product, we have

$$D = \lim_{t \to \infty} -\frac{(2\pi)^{-2}}{t} \log \left| \frac{\mu([a, b] \times \mathbb{T} \mid \Omega; t)}{b - a} - 1 \right| \tag{2.22}$$

and for $a = j$, $b = j + dj$ in (2.22) we have exactly the integrated density -1, within the bars.

For an area-preserving map the corresponding measure is given by

$$\mu(B \mid \Omega; n) = \frac{\mu_L(B \cap M^n \Omega)}{\mu_L(\Omega)} \tag{2.23}$$

As a consequence the moments in its case are given by

$$\begin{aligned} \mathbf{E}_\Omega(j_n^q) &= \frac{1}{\mu_L(\Omega)} \int_{\mathbb{T}^2} dj_0\, d\theta_0\, j_n^q \mathscr{X}_\Omega(j_0, \theta_0) \\ &= \frac{1}{\mu_L(\Omega)} \int_{\mathbb{T}^2} dj\, d\theta \quad j^q \mathscr{X}_{M^n(\Omega)}(j, \theta) = \int_{\mathbb{T}^2} j^q\, d\mu(j, \theta \mid \Omega; n) \end{aligned} \tag{2.24}$$

where $j_n = [M^n(j_0, \theta_0)]_j$ and the change of variables $(j, \theta) = M^n(j_0, \theta_0)$ is made in the second step using the invariance of the Lebesgue measure under the action of the map M. With this definition of the moments the diffusion coefficient is given by (2.12). The existence of the limit is related to a convenient decay of the correlation function of j^q and $\mathscr{X}_\Omega(\theta, j)$. If the limits exist for all moments, then a weak convergence of the measure $\mu(B \mid \Omega; n)$ to $\mu_L(B)$ as $n \to \infty$ is implied. The convergence is exponentially fast, too, namely for a large choice of the sets B and Ω the correlation

$$|\mu_L(B \cap M^n(\Omega)) - \mu_L(B) \cdot \mu_L(\Omega)| \tag{2.25}$$

must have an exponential decay. As a consequence we can write (2.22) simply by changing t into n.

A well-known result in ergodic theory states that the convergence in n as $n \to +\infty$ of the correlations (2.25) cannot be uniform with respect to B and Ω. The existence of the above limit (2.22) is suggested by numerical evidence; moreover, for the one-dimensional map it has been proved that for a large choice of subsets of $[0, 1[$ the limit exists and the decay is exponential (see Appendix A).

3. EXISTENCE OF $D_2(\Omega)$ FOR THE AUTOMORPHISM OF THE TORUS

In this section we state the main result concerning the existence of $D_{21}(\Omega)$ for a particular mapping M, the toral automorphism. We prove that the diffusion coefficient $D_2(\Omega)$ corresponding to a uniform initial density $\rho(j, \theta; 0) = \mathscr{X}_\Omega(j, \theta)/\mu_L(\Omega)$, where $\Omega \subset \mathbb{T}^2$ is a parallelogram on the torus, exists for almost all such sets and has the same value. The proof is given by using the explicit expressions for the moments $\mathbf{E}_\Omega(j_n^q)$, which have been written also when Ω is an arbitrary polygon.

The map M we consider is the algebraic automorphism of the torus $\mathbb{T}^2$, known also as "Arnold's cat map":

$$M\begin{pmatrix}\theta\\ j\end{pmatrix} = \begin{pmatrix}K+1 & 1\\ K & 1\end{pmatrix}\begin{pmatrix}\theta\\ j\end{pmatrix} \quad \text{mod}\ \mathbb{T}^2 \tag{3.1}$$

for $K \in \mathbb{Z}\backslash\{0\}$; $|K+2| > 2$ and $(\theta, j) \in \mathbb{T}^2$. We denote by λ the largest eigenvalue of the map, so that $\log \lambda$ is the positive Lyapunov exponent

$$\lambda = \frac{2 + K + [K(K+4)]^{1/2}}{2} \tag{3.2}$$

Although this mapping has good ergodic and statistical properties, the exact evaluation of $D_2(\Omega)$ is rather complicated. Also the numerical computation of the moments $\mathbf{E}_\Omega(j_n^q)$ involves some difficulties because the hyperbolic character of the map implies a fast loss of information. For instance, if $K = 1$, then the largest eigenvalue $(3 + \sqrt{5})/2$ is bigger than 2 and the loss of accuracy is more than one bit per iteration; with ordinary accuracy ~ 50 bits the number of meaningful iterations is less than 50 and iterating 2000 times, as we did, demands a number of significant digits at least 40 times higher.

In this respect an analytic result is relevant to guide the numerical investigations; for almost all initial sets Ω belonging to the family of parallelograms the coefficient $D_2(\Omega)$ exists and is $2 \log \lambda\, (2\pi)^{-2}$. This is not surprising if we keep in mind that $D_2(\Omega)$ is defined by (2.12); in fact the argument of the logarithm is the correlation between the two functions $F_1(\theta, j) = j^p$ and $F_2(\theta, j) = \mathscr{X}_\Omega(\theta, j)$. When the functions F_1, F_2 are sufficiently regular (for example, Hölder continuous), then an upper bound of the form λ^{-2n} to the correlation $|\mathbf{E}_\Omega(j_n^2) - \mathbf{E}_\Omega(j_\infty^2)|$ can be given (indeed λ^{-2} is the eigenvalue of the Perron–Frobenius operator closest to 1). Whenever the functions F_1 and F_2 are "piecewise-Hölder" on finitely many subdomains M_i with piecewise smooth boundary and whose union is the torus (and the observable F_2 belongs to this class), then the symbolic dynamics technique can be applied as well, as first pointed out in ref. 12. However, in this case and for our system, the dominant term of the decay of correlations obtained from symbolic dynamics is given by the total measure of the elements of the (iterated) Markov partition at "time n" intersecting the boundaries of the M_i and therefore the upper bound to the decay of correlations is λ^{-n} rather than λ^{-2n}. This upper bound for the correlation decay is considerably improved by our main theorem, which gives again λ^{-2n}. A lower bound λ^{-2n} is also proved from which the existence of the limit defining $D_2(\Omega)$ easily follows. One could notice that if Ω itself is a Markovian rectangle, the (local) Hölder property is automatically verified inside it and we recover an upper bound λ^{-2n} to the correlation decay.

The nonexistence of the limit for some sets Ω can be understood by considering the one-dimensional map $M(x) = 2x \bmod 1$. We define the diffusion coefficient $D_2(\Omega)$ as in (2.12), replacing M by M^{-1}. Letting Ω be any finite or countable disjoint union of closed intervals in $[0, 1[$, it is easy to produce examples of domains Ω for which $D_2(\Omega)$ is ∞, and the decay of correlations is faster than exponential. We refer to Appendix A for the precise statement of this result.

The technique we use relies on the geometric properties of parallelograms and involves two steps: first we give an explicit expression for the correlation integral, and second we allow the movement of the vertices of the parallelogram in a subset of Lebesgue measure arbitrarily close to 1 in order to prove the existence of the desired limit. This result obtained for a large class, in a measure-theoretic sense, of parallelograms and for a density function shows that our definition of the diffusion coefficient is meaningful.

Letting $\mathbf{x} = (\theta, j) \in \mathbb{T}^2$, we consider two points $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{T}^2$ on the torus and $\Delta\mathbf{x} \in \mathbb{T}^2$ in order to build the parallelogram whose vertices are given by

$$\mathbf{x}_1, \quad \mathbf{x}_1 + \Delta\mathbf{x}, \quad \mathbf{x}_2, \quad \mathbf{x}_2 + \Delta\mathbf{x} \tag{3.3}$$

We consider then two sets $A \subseteq \mathbb{T}^2$ and $B(\mathbf{x}_2) \subseteq \mathbf{T}^2$ of measure 1:

$$\mu_L(A) = 1, \qquad \mu_L(B(\mathbf{x}_2)) = 1 \tag{3.4}$$

and a family of parallelograms obtained by choosing $\Delta\mathbf{x} \in A$ and $\mathbf{x}_1 \in B(\mathbf{x}_2)$. This means that for any given vertex $\mathbf{x}_2$ one can choose the other two free vertices on μ_L-almost any point of the torus.

Theorem. There exists a μ_L-measurable set $A \subseteq \mathbb{T}^2$, with $\mu_L(A) = 1$, and for every $\mathbf{x}_2 \in \mathbb{T}^2$ a μ_L-measurable set $B(\mathbf{x}_2) \subseteq \mathbb{T}^2$, with $\mu_L(B(\mathbf{x}_2)) = 1$, such that $\forall \Delta\mathbf{x} \in A$ and $\mathbf{x}_1 \in B(\mathbf{x}_2)$, the limit defining the diffusion coefficient on the parallelogram (3.3), with vertices in the covering plane, exists and is twice the Lyapunov exponent times $1/4\pi^2$:

$$D_2(\Omega) = \lim_{n \to +\infty} -\frac{1}{4\pi^2 n} \log \left| \int_{\mathbb{T}^2} j^2 \, d\mu(j, \theta \mid \Omega; n) - \int_{\mathbb{T}^2} j^2 \, d\mu(j, \theta \mid \Omega; \infty) \right|$$

$$= 2 \log \lambda \cdot \frac{1}{4\pi^2} \tag{3.5}$$

where $d\mu(j, \theta \mid \Omega; \infty) = d\mu_L(j, \theta)/\mu_L(\mathbb{T}^2)$—the factor $1/4\pi^2$ will be omitted from now on.

The previous statement shows that for a random choice of the parameters $\mathbf{x}_1$, $\mathbf{x}_2$, $\Delta\mathbf{x}$ the probability of getting a parallelogram for which the diffusion coefficient exists and is equal to $2 \log \lambda$ is 1. The proof of the theorem is lengthy and is described in Appendix B. We also notice that the same ideas of the proof can be applied to even moments of higher order and, more generally, to continuous functions of the only variable j.

4. NUMERICAL RESULTS AND CONCLUSIONS

The result of the main theorem has been numerically checked and more generally the existence of the limit (2.12) defining $D_q(\Omega)$ has been

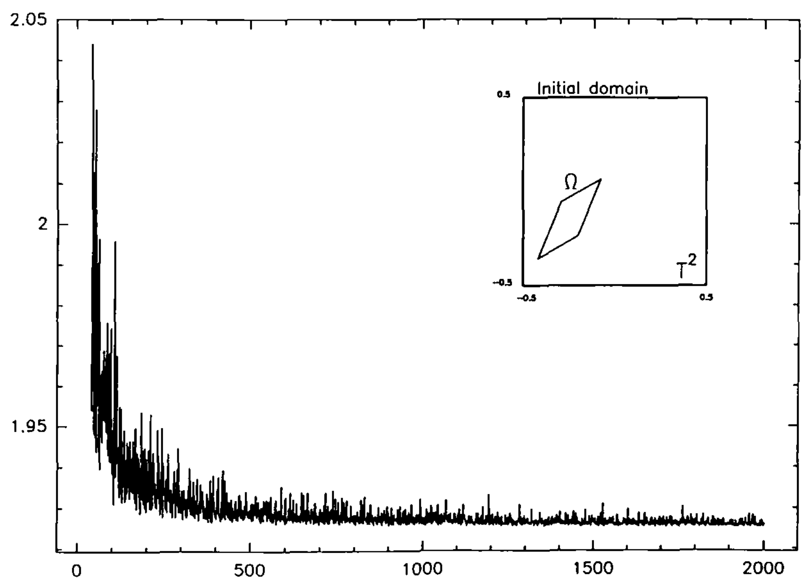

Fig. 1. Values of $-(1/n) \log |\mathbf{E}_\Omega(j_n^2) - \mathbf{E}_\Omega(j_\infty^2)|$ versus n for an initial domain consisting of a parallelogram. For the cat map the Lyapunov exponent is ~ 0.9624.

tested for values of q different from 2 and for arbitrary polygonal domains Ω. The existence of the limit $D_q(\Omega)$ related to the integrated density has also been checked and it has been found that for any choice of Ω all the limits have the same value $D_q(\Omega) = 2\log\lambda$. The explicit expression of $\mathbf{E}_\Omega(j_n^q)$ used in the numerical calculations is given by formulas (B.2)–(B.5) of Appendix B. The limit was checked with a good accuracy, within 0.1% and 0.3%, by considering typically $N = 2000$ iterations, which were obtained by workinig with a number of significant digits at least 50 times higher with respect to ordinary floating point representation. We report here some results for the Arnold's cat map (3.1) with $K = 1$ for the following choices of Ω: a parallelogram, a hexagon, and a pentagon. Using the explicit relations for polygonal domains, the correlations $\mathbf{E}_\Omega(j_n^q)$ can be computed with high accuracy and decay, for large n, in agreement with the result of the theorem (see Figs. 1–3). For the second moment the following algorithm was used when Ω is a parallelogram of vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3$ and sides $\boldsymbol{\xi} = \mathbf{x}_1 - \mathbf{x}_0 = (\xi_\theta, \xi_j)$, $\boldsymbol{\eta} = \mathbf{x}_2 - \mathbf{x}_1 = (\eta_\theta, \eta_j)$:

$$\begin{aligned}\mathbf{E}_\Omega(j_n^2) - \mathbf{E}_\Omega(j_\infty^2) = {} & \frac{1}{a\lambda^{2n} + b + c\lambda^{-2n}} \\ & \times [\Theta_2 \circ \Pi_j \circ \Phi^n(\mathbf{x}_1) + \Theta_2 \circ \Pi_j \circ \Phi^n(\mathbf{x}_3) \\ & - \Theta_2 \circ \Pi_j \circ \Phi^n(x_2) - \Theta_2 \circ \Pi_j \circ \Phi^n(\mathbf{x}_0)] \qquad (4.1)\end{aligned}$$

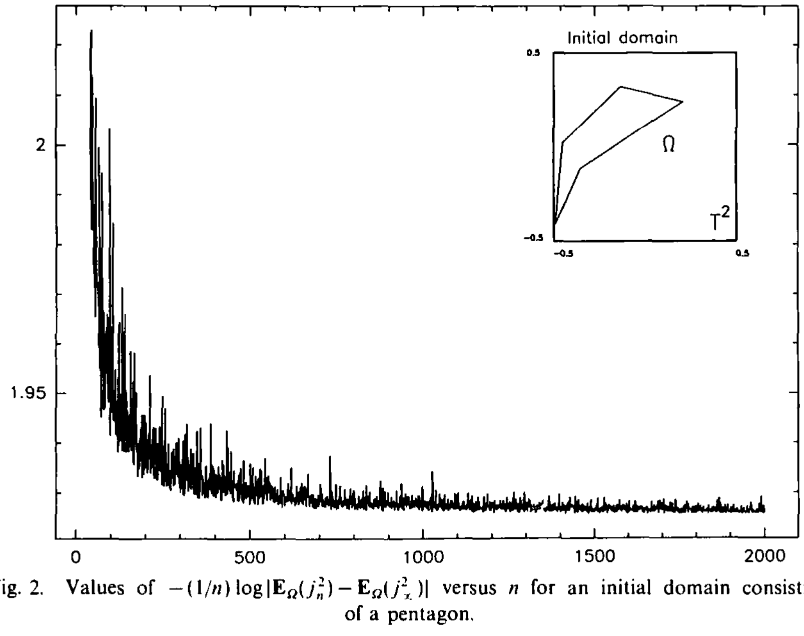

Fig. 2. Values of $-(1/n)\log|\mathbf{E}_\Omega(j_n^2) - \mathbf{E}_\Omega(j_\infty^2)|$ versus n for an initial domain consisting of a pentagon.

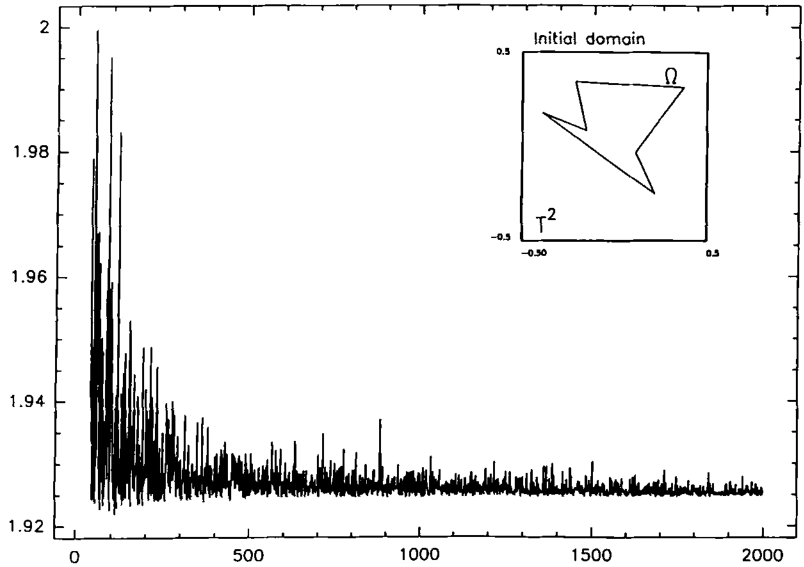

Fig. 3. Values of $-(1/n)\log|\mathbf{E}_\Omega(j_n^2)-\mathbf{E}_\Omega(j_\infty^2)|$ versus n for an initial domain consisting of a hexagon.

where

$$\begin{aligned} a &= -\frac{1}{(\lambda-\lambda^{-1})^2}(K\xi_0+\xi_j-\xi_j\lambda^{-1})(K\eta_0+\eta_j-\eta_j\lambda^{-1}) \\ b &= \frac{1}{(\lambda-\lambda^{-1})^2}[(K\xi_0+\xi_j-\xi_j\lambda^{-1})(K\eta_0+\eta_j-\eta_j\lambda) \\ &\quad + (K\eta_0+\eta_j-\eta_j\lambda^{-1})(K\xi_0+\xi_j-\xi_j\lambda)] \\ c &= -\frac{1}{(\lambda-\lambda^{-1})^2}(K\xi_0+\xi_j-\xi_j\lambda)(K\eta_0+\eta_j-\eta_j\lambda) \end{aligned} \tag{4.2}$$

and the functions Θ_2 and Π_j are defined in Appendix B.

One can observe that if the constant a in (4.1) is nonzero, then the moment can be written as λ^{-2n} times an expression whose logarithm divided by n converges to zero as specified in the previous theorem.

From the computational point of view the exponential growth λ^n of the map M requires an accuracy of at least λ^{-N} if N iterations have to be computed. Integer arithmetic was used representing the initial conditions with at least $N\log_{10}\lambda$ digits.

The dependence of $D_q(\Omega)$ on q was also checked. For $q=3, 4, 6$ the same value $2\log\lambda$ as for $q=2$ was found within the numerical errors, which were typically less than 1% for $n\leqslant 2000$. The limit (2.22) for the integrated density was also considered and it was still found that $D_q(\Omega)=2\log\lambda$ for a variety of polygonal domains.

These results suggest that the moments are the same as for a diffusion equation, at least for a large choice of initial domains Ω, and we are led to make the following conjecture. For almost all the initial domains Ω the limits $D_q(\Omega)$ exist and have the same value. The numerical method we used can be easily extended with the same accuracy to correlation integrals of arbitrary functions of the variable j, integrable with respect to the Lebesgue measure on $\mathbb{R}$. Therefore it can be used as a test in order to verify the accuracy of other methods to compute correlation integrals. The next step of the work presented here is its generalization to other, less trivial automorphisms of the torus, such as almost hyperbolic mappings, or mappings with singularities, including billiards. In this case, some good ergodic properties persist, but the decay of correlations would need a careful and deeper investigation.

APPENDIX A

We consider here the endomorphism T of the one-dimensional torus $[0,1[$ onto itself defined by $T(x)=2x \bmod [0,1[$, $\forall x\in[0,1[$, and compute the limit

$$\int_0^1 y^2\mathscr{X}_{T^{-n}(\Omega)}(y)\,d\mu_L(y)-\int_0^1 y^2\,d\mu_L(y)\cdot\mu_L(\Omega) \tag{A.1}$$

We have then[13] the following proposition.

Proposition. For a given Markov interval the limit exists and takes the value log 2, corresponding to the Lyapunov exponent of the map.

But if we consider a countable union of intervals $\Omega=\bigcup_{i=1}^{\infty}[a_i,b_i]$, with $a_i<b_i<a_{i+1}$, $\forall i\in\mathbb{N}$, then[13]:

(i) If the condition

$$\sum_{i=1}^{\infty}(b_i^2-a_i^2)\neq\sum_{i=1}^{\infty}(b_i-a_i) \tag{A.2}$$

is satisfied, the limit exists and takes the value log 2.

(ii) If on the contrary there holds

$$\sum_{i=1}^{\infty}(b_i^2-a_i^2)=\sum_{i=1}^{\infty}(b_i-a_i) \tag{A.3a}$$

together with the further constraint

$$\sum_{i=1}^{\infty} \frac{b_i - a_i}{6} + \sum_{i=1}^{\infty} \frac{b_i^3 - a_i^3}{3} - \sum_{i=1}^{\infty} \frac{b_i^2 - a_i^2}{2} \neq 0 \tag{A.3b}$$

the limit also exists, but its value is 2 log 2.

(iii) If finally both conditions (A.2) and (A.3b) are not satisfied, so that

$$\begin{cases} \sum_{i=1}^{\infty} (b_i^2 - a_i^2) = \sum_{i=1}^{\infty} (b_i - a_i) \\ \sum_{i=1}^{\infty} (b_i^3 - a_i^3) = \sum_{i=1}^{\infty} (b_i - a_i) \end{cases} \tag{A.4}$$

the limit is not finite. ▮

The result shows that for a large choice of initial domains the limit (A.1) exists and takes the value log 2. In particular, in the case of finite unions $\bigcup_{i=1}^{n} [a_i, b_i]$ of intervals the constraints $a_i < b_i < a_{i+1}$, $\forall i = 1,..., n$, define a nonempty open simplex of the rectangle $[0, 1[^n$, with positive Lebesgue measure on the same space. The undesired condition corresponding to (A.3a) describes an algebraic surface of codimension one and vanishing Lebesgue measure on $[0, 1[^n$. In this sense (A.3a) occurs with probability zero.

APPENDIX B

In this appendix[4] we sketch the proof of the Theorem. We first denote the nth iterate of the tangent matrix Φ of (3.1) by

$$\Phi_n \stackrel{\text{def}}{=} \begin{pmatrix} K+1 & 1 \\ K & 1 \end{pmatrix}^n \stackrel{\text{def}}{=} \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix} \tag{B.1}$$

and the largest eigenvalue of Φ by $\lambda = [K + 2 + (K^2 + 4K)]^{1/2}/2$, so that $\lambda_L = \log \lambda$. Moreover, $\forall q \in \mathbb{N}$ we define the function Θ: $\mathbb{R} \to \mathbb{R}$ given by

$$\Theta_{2q}(y) \stackrel{\text{def}}{=} \frac{y^{2q+2}}{(2q+2)(2q+1)} - \mathbf{E}_\Omega(y_\infty^{2q}) \cdot \frac{y^2}{2} \qquad \forall y \in [-1/2, 1/2[\tag{B.2}$$

[4] From now on we pose $(\theta, j) = (x, y)$.

where $\mathbf{E}_\Omega(y_\infty^{2q}) = 1/[2^{2q}(2q+1)]$, and

$$\Theta_{2q-1}(y) \stackrel{\text{def}}{=} \frac{y^{2q+1}}{(2q+1)\,2q} - \frac{1}{(2q+1)\,2q\cdot 2^{2q}}\cdot y \qquad \forall y \in [-1/2, 1/2[\tag{B.3}$$

both periodically continued with period 1 on $\mathbb{R}$.

We have then the following lemma, whose proof relies on an application of the Gauss–Green formula:

Lemma B.1. Let $\Omega \subset \mathbb{T}^2$ be a polygon with p sides whose vertices in the covering plane are numbered counterclockwise, with coordinates $(x_j, y_j) \in \mathbb{T}^2$, $\forall j = 0, 1,..., p-1$, $(x_p, y_p) \equiv (x_0, y_0)$. Then the moment of order m is given by

$$\begin{aligned}\mathbf{E}_\Omega(y_n^m) &= \mathbf{E}_\Omega(y_\infty^m) + \frac{1}{\mu_L(\Omega)c_n}\sum_{q=0}^{p-1} \Psi_m(c_n x_{q+1} + d_n y_{q+1}; c_n x_q + d_n y_q) \\ &\quad \cdot (y_{q+1} - y_q) \\ &= \mathbf{E}_\Omega(y_\infty^m) - \frac{1}{\mu_L(\Omega)d_n}\sum_{q=0}^{p-1} \Psi_m(c_n x_{q+1} + d_n y_{q+1}; c_n x_q + d_n y_q) \\ &\quad \cdot (x_{q+1} - x_q)\end{aligned} \tag{B.4}$$

where we set, $\forall \alpha, \beta \in \mathbb{R}$,

$$\Psi_m(\alpha; \beta) := \begin{cases} \dfrac{\Theta_m(\alpha) - \Theta_m(\beta)}{\alpha - \beta} & \text{if } \alpha \neq \beta \\ \Theta'_m(\alpha) & \text{if } \alpha = \beta \end{cases} \quad \blacksquare \tag{B.5}$$

Remark B.2. The above lemma can be generalized to a countable union of disjoint polygons $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$, with $\Omega_i \subset \mathbb{T}^2$, and by replacing the observable y^m with any other function $g(x, y) = g(y) \in L^1(\mathbb{T}^2)$.[13]

The existence of the limit (2.12) will be proved by taking Ω as a parallelogram. In this case the expression (B.4) for the moment of order m can be rewritten in a more compact form.

Lemma B.3. By defining Π_j as the linear projection along the j axis in $\mathbb{R}^2$, for a parallelogram Ω of vertices $\mathbf{x}_0$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3 \in \mathbb{T}_2$ and sides $\boldsymbol{\xi} = \mathbf{x}_1 - \mathbf{x}_0 = (\xi_\theta, \xi_j)$, $\boldsymbol{\eta} = \mathbf{x}_2 - \mathbf{x}_1 = (\eta_\theta, \eta_j)$, the moment of order $m \in \mathbb{N}$ at time $n \in \mathbb{N}$ is given by

$$\begin{aligned}\mathbf{E}_\Omega(y_n^m) = \mathbf{E}_\Omega(y_\infty^m) &- \frac{1}{(c_n\xi_\theta + d_n\xi_j)(c_n\eta_\theta + d_n\eta_j)} \\ &\times [\Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_1) + \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_3) \\ &- \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_2) - \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_0)] \quad \blacksquare\end{aligned} \tag{B.6}$$

We now confine ourselves to the second moment, by introducing —for simplicity's sake—the function $\Theta(y) = \Theta_2(y) = y^4/12 - y^2/24$, $\forall y \in [-1/2, 1/2[$, which coincides with Θ_2, up to an additive constant not involved in (B.6). Let us further define a parameter $t \in [-\frac{1}{2}, \frac{1}{2}[$, and, for fixed t, the function

$$\Delta_t: \quad [-\tfrac{1}{2}, \tfrac{1}{2}[\xrightarrow[\text{onto}]{1\cdot 1} \mathbb{R}$$

given by

$$\Delta_t(y) \stackrel{\text{def}}{=} \Theta(y+t) - \Theta(y) = \Theta(y + t \bmod [-\tfrac{1}{2}, \tfrac{1}{2}[) - \Theta(y), \qquad y \in [-\tfrac{1}{2}, \tfrac{1}{2}[\tag{B.7}$$

which can be periodically continued—with period 1—on $\mathbb{R}$ as a C^2 function, as a difference of C^2 functions. Apart from the case $t = 0$, where the function $\Delta_t(y)$ is trivially constant, equal to zero, a close analysis of the graph of Δ_t (Fig. 4) leads to the following lemma, which is crucial in the proof of the Theorem.

Lemma B.4. For any fixed $t \in [-\frac{1}{2}, \frac{1}{2}[$, $t \neq 0$, any interval $I \subset \mathbb{R}$ of given length $\varepsilon > 0$ such that $\varepsilon < (9/2^{12})|t|$ has an inverse image through Δ_t in $[-\frac{1}{2}, \frac{1}{2}[$, $\Delta_t^{-1}(I)$, whose Lebesgue measure admits the upper bound $\mu_L(\Delta_t^{-1}(I)) \leqslant 8(\varepsilon/|t|)^{1/2}$. ▮

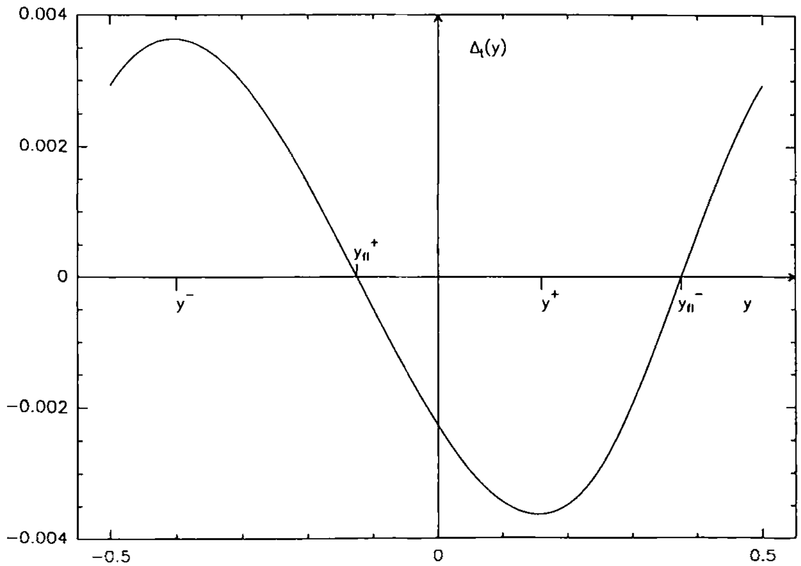

Fig. 4. Graph of the function $\Delta_t(y)$ for $t = 0.25$.

We are now ready to formulate the main lemma.

Lemma B.5. $\forall \delta \in]0,1[$, $\forall \mathbf{x}_2 \in \mathbb{T}^2$, and $\forall \delta' < \bar{\delta}$, with $\bar{\delta} \in]0,1[$ fixed, $\exists$ a Lebesgue-measurable set $B(\delta) \subseteq \mathbb{T}^2$ of measure $\mu_L(B(\delta)) \geqslant 1-\delta$ and $\exists$ a Lebesgue-measurable set $B'(\delta, \mathbf{x}_2, \delta') \subseteq \mathbb{T}^2$ of measure $\mu_L(B'(\delta, \mathbf{x}_2, \delta')) \geqslant 1-\delta'$ such that $\forall \Delta\mathbf{x} \in B(\delta)$ and $\forall \mathbf{x}_1 \in B'(\delta, \mathbf{x}_2, \delta')$ there holds

$$\exists \lim_{n \to +\infty} -\frac{1}{n} \log |\Theta_2[\Pi_j \circ \Phi^n(\mathbf{x}_1 + \Delta\mathbf{x})] - \Theta_2[\Pi_j \circ \Phi^n(\mathbf{x}_1)] - \Theta_2[\Pi_j \circ \Phi^n(\mathbf{x}_2 + \Delta\mathbf{x})] + \Theta_2[\Pi_j \circ \Phi^n(\mathbf{x}_2)]| = 0 \quad \blacksquare \tag{B.8}$$

Before giving the proof of the lemma, we want to point out the geometrical meaning of the result.

Remark B.6. Notice that whenever no one of the vectors $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ lies along the stable direction then, for large values of $n \in \mathbb{N}$, the denominator in (B.6) has the asymptotic behavior $(c_n \xi_\theta + d_n \xi_j)(c_n \eta_\theta + d_n \eta_j) \sim \lambda^{2n}$. On the contrary, if one—and only one—of those vectors is oriented along the stable direction, the previous asymptotic relation is replaced by $(c_n \xi_\theta + d_n \xi_j)(c_n \eta_\theta + d_n \eta_j) \sim 1$. In the first case the diffusion coefficient will be expressed by the following limit:

$$\begin{aligned} -\lim_{n \to +\infty} \frac{1}{n} \log |\mathbf{E}_\Omega(y_n^m) - \mathbf{E}_\Omega(y_\infty^m)| \\ = 2 \log |\lambda| - \lim_{n \to +\infty} \frac{1}{n} \log |\Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_1) + \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_3) \\ - \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_2) - \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_0)| \end{aligned} \tag{B.9}$$

whereas in the second case we have

$$\begin{aligned} -\lim_{n \to +\infty} \frac{1}{n} \log |\mathbf{E}_\Omega(y_n^m) - \mathbf{E}_\Omega(y_\infty^m)| \\ = -\lim_{n \to +\infty} \frac{1}{n} \log |\Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_1) + \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_3) \\ - \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_2) - \Theta_m \circ \Pi_j \circ \Phi^n(\mathbf{x}_0)| \end{aligned} \tag{B.10}$$

The statement of Lemma B.5 guarantees that for a random choice of the parameters $\Delta\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$ the probability that the diffusion coefficient exists for the parallelogram of vertices $\mathbf{x}_0 = \mathbf{x}_1 + \Delta\mathbf{x}$, $\mathbf{x}_1$, $\mathbf{x}_2$, $\mathbf{x}_3 = \mathbf{x}_2 + \Delta\mathbf{x}$ and takes the value $2 \log |\lambda|$ is arbitrarily close to one. We point out that the set of

$\Delta\mathbf{x} \in B(\delta)$ along the stable direction has zero Lebesgue measure; moreover, for fixed $\mathbf{x}_2$, the set of $\mathbf{x}_1 \in B'(\delta, \mathbf{x}_2, \delta')$ such that the vector $\mathbf{x}_2 - \mathbf{x}_1$ is along the stable direction is in turn of zero measure.

Proof of Lemma B.2. With the previous notation, let us define the following quantity, which is a particular value of the parameter t introduced in (B.7) especially adapted to study the function in the logarithm of (B.8):

$$\begin{aligned} t &\stackrel{\text{def}}{=} \Pi_j \circ \Phi^n(\Delta\mathbf{x}) \bmod [-1/2, 1/2[\\ &= c_n \cdot \Delta x + d_n \cdot \Delta y \bmod [-1/2, 1/2[\\ &= \Pi_j \circ M^n(\Delta\mathbf{x}) \bmod [-1/2, 1/2[\in [-1/2, 1/2[\end{aligned} \tag{B.11}$$

We will require that $|t| > \varepsilon_n$, where $\varepsilon_n = \nu/n^k$, with $\nu > 0$ and $k > 1$ constants which will be determined in the following. This prescription will follow from a choice of $\Delta\mathbf{x} =^{\text{def}} (\Delta x, \Delta y)$.

We also define $\varepsilon'_n = \nu'/n^{k'}$, where con $\nu' > 0$ and $k' > 1$ will be determined in such a way that

$$2\varepsilon'_n < \frac{9}{2^{12}} \varepsilon_n \qquad \forall n \in \mathbb{N} \tag{B.12}$$

Let us define the set

$$C \stackrel{\text{def}}{=} C(k, \nu) = \bigcup_{n=1}^{\infty} M^{-n}(\{(a, b) \in \mathbf{T}^2 / a \in [-\tfrac{1}{2}, \tfrac{1}{2}[\text{ and } |b| < \varepsilon_n\}) \subset \mathbb{T}^2 \tag{B.13}$$

It is easy to see that $\mu_L(C) \leqslant \sum_{n=1}^{\infty} \varepsilon_n$.

We now introduce the complementary set $B =^{\text{def}} \mathbb{T}^2 \backslash C$, and assuming that $\Delta\mathbf{x} \in B$, we have that

$$\forall n \in \mathbb{N}: \quad t = \Pi_j \circ M^n(\Delta\mathbf{x}) = c_n\, \Delta x + d_b\, \Delta y \bmod [-\tfrac{1}{2}, \tfrac{1}{2}[$$

satisfies the condition $|t| > \varepsilon_n$.

We fix $\mathbf{x}_2 \in \mathbb{T}^2$ and define $E_n = \Theta[\Pi_j \circ M^n(\mathbf{x}_2 + \Delta\mathbf{x})] - \Theta[\Pi_j \circ M^n(\mathbf{x}_2)]$. It is easy to see that the condition

$$E_n - \varepsilon'_n < \Theta[\Pi_j \circ M^n(\mathbf{x}_1 + \Delta\mathbf{x})] - \Theta[\Pi_j \circ M^n(\mathbf{x}_1)] < E_n + \varepsilon'_n \tag{B.14}$$

will take place for a set of values of $\mathbf{x}_1 \in \mathbb{T}^2$ of measure at most

$$8\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{1/2} \tag{B.15}$$

This set must be discarded in the choice of the initial datum $\mathbf{x}_1$. We now define the set $C' \subset \mathbb{T}^2$, which is the union over $n \in \mathbb{N}$ of the sets of points satisfying condition (B.14). The measure of this set is clearly bounded by

$$\mu_L(C') \leqslant \sum_{n=1}^{\infty} 8\left(\frac{2\varepsilon'_n}{\varepsilon_n}\right)^{1/2}$$

The set C' depends on the sequence ε_n, i.e., on the parameters $v > 0$ and $k > 1$, on the choice of $\mathbf{x}_2 \in \mathbb{T}^2$, and finally on the sequence ε'_n—namely the parameters $v' > 0$ and $k' > 1$—satisfying the condition (B.12).

If we now introduce the completementary set $B' =^{\text{def}} \mathbb{T}^2 \backslash C'$ and arbitrarily fix $\mathbf{x}_1 \in B'$, we have that $\forall n \in \mathbb{N}$ the following bound holds:

$$|\Theta[\Pi_j \circ M^n(\mathbf{x}_1 + \Delta\mathbf{x})] - \Theta[\Pi_j \circ M^n(\mathbf{x}_1)] - E_n| \geqslant \varepsilon'_n \tag{B.16}$$

Since by the periodicity of the function Θ we can replace M with Φ in the preceding expression, the left-hand side of (B.16) coincides with the argument of the logarithm in (B.8). This argument, which also admits a trivial upper bound, can go to zero s $n \to +\infty$ slower than $v'/n^{k'}$, so that the limit in (B.8) will be zero.

We now show that be a suitable choice of the parameters k, k', v, v' the measure of the sets C and C' can be made arbitrarily small.

The condition (B.12) can be rewritten as $2(v'/n^{k'}) < (9/2^{12})\ v/n^k$, $\forall n \in \mathbb{N}$, so that (B.15) will take the form $8[(2v'/n^{k'})\, n^k/v]^{1/2}$. Then we simply have to take

$$k \geqslant 2, \qquad k' \geqslant k+4, \qquad v' < \frac{9}{2^{12}}\, v, \qquad v < \frac{6}{\pi^2} \tag{B.17}$$

in order to get that:

(i) (B.15) reads

$$8\left(\frac{2v'}{v}\right)^{1/2} \frac{1}{n^{(k'-k)/2}} < 8\left(\frac{2v'}{v}\right)^{1/2} \frac{1}{n^2}, \qquad \forall n \in \mathbb{N}$$

(ii) The measure of the best B is bounded by

$$\mu_L(B) \geqslant 1 - v \sum_{n=1}^{\infty} \frac{1}{n^k} > 0$$

(iii) The measure of the set B' admits the lower bound

$$\mu_L(B') \geqslant 1 - 8\left(\frac{2v'}{v}\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{n^{(k'-k)/2}} > 0 \tag{B.18}$$

If we now fix the values of k, k' and choose v and v' small enough according to (B.17), we can construct two sets B and B' of measure arbitrarily close to 1 as

$$\mu_L(B) \geqslant 1 - v \sum_{n=1}^{\infty} \frac{1}{n^k} = 1 - \delta \tag{B.19}$$

and

$$\mu_L(B') \geqslant 1 - 8\left(\frac{2v'}{v}\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{n^{(k'-k)/2}} = 1 - \delta'$$

where we have defined

$$\delta = v \sum_{n=1}^{\infty} \frac{1}{n^k} \qquad \text{and} \qquad \delta' = 8\left(\frac{2v'}{v}\right)^{1/2} \sum_{n=1}^{\infty} \frac{1}{n^{(k'-k)/2}}$$

For example, we can choose $k = 2$ and $k' = 6$ so that, because of $v' < 9v/2^{12}$, δ can assume any value in the interval $]0, 1[$, while owing to the bound $v < 6/\pi^2$, δ' will take any value in $]0, \sqrt{2}\pi^2/2^4[$. On the contrary, fixing $\delta \in]0, 1[$ and $\delta' \in]0, \sqrt{2}\pi^2/2^4[$ is equivalent to determining uniquely v and v' in accordance with (B.17), and therefore the sequences ε_n and ε'_n. As a conclusion we can write that $\forall \delta \in]0, 1[$ and $\forall \delta' \in]0, \sqrt{2}\pi^2/2^4[$ there exists a set B, dependent on the choice of δ: $B = B(\delta)$, of measure $\geqslant 1 - \delta$, and a set B', dependent on δ, $\mathbf{x}_2 \in \mathbb{T}^2$, and δ': $B' = B'(\delta, \mathbf{x}_2, \delta')$, of measure $\geqslant 1 - \delta'$, which completes the proof. ▌

Proof of the Main Theorem. Let us consider a decreasing sequence $(\delta_n)_{n \in \mathbb{N}}$ such that $\delta_n \in]0, 1[$ $\forall n \in \mathbb{N}$ and $\lim_{n \to +\infty} \delta_n = 0$. For every $n \in \mathbb{N}$ let $(\delta'_{n,m})_{m \in \mathbb{N}}$ be a decreasing sequence on $]0, 1[$, with $\lim_{m \to +\infty} \delta'_{n,m} = 0$ and $\delta'_{n,1}$ small enough—say, $\delta_n > \delta'_{n,1} > \delta'_{n,m}$, $\forall m \in \mathbb{N}$. The main theorem follows from Lemma B.5 by posing

$$A := \bigcup_{n=1}^{\infty} B(\delta_n) \tag{B.20}$$

and, for fixed $\mathbf{x}_2 \in \mathbb{T}^2$,

$$B(\mathbf{x}_2) := \bigcap_{n=1}^{\infty} \bigcup_{m=1}^{\infty} B'(\delta_n, \mathbf{x}_2, \delta'_{n,m}) \tag{B.21}$$

as it is straightforward to verify that both A and $B(\mathbf{x}_2)$ are μ_L-measurable and that $\mu_L(A) = \mu_L(B(\mathbf{x}_2)) = 1$. ▌

ACKNOWLEDGMENTS

We thank A. Bazzani and S. Isola for useful discussions on the statistical motivation of the problem. S.S. was supported by an MPI 60% grant, G.T. and S.V. by NATO grant 890383, and S.V. by contract CEE SC1*0281.

REFERENCES

1. B. V. Chirikov, M. A. Lieberman, D. L. Shepelyansky, and E. Vivaldi, A theory of modulational diffusion, *Physica* **14D**:289 (1985).
2. I. Dana, Hamiltonian transport of unstable periodic orbits. *Physica* **39D**:205–230 (1989).
3. V. Room-Kedar and S. Wiggins, Transport on two dimensional maps, *Arch. Rat. Mech. Anal.* **109**:239 (1990).
4. P. Holmes and J. Marsden, Horseshowes and Arnold diffusion for Hamiltonian systems on Lie groups, *Indiana Univ. Math. J.* **32**:2 (1983).
5. P. Lochak, Effective speed of Arnold diffusion and small denominators, *Phys. Lett. A* **143**:39 (1990).
6. J. Meiss and E. Ott, Markov tree model of intrinsic transport in Hamiltonian systems, *Phys. Rev. Lett.* **55**:2741 (1985).
7. D. L. Bruhwiler and R. Cary, Diffusion of particles in a slowly modulated wave, *Physica* **40D**:265–282 (1989).
8. I. S. Aronson, M. I. Rabinovich, and L. Sh. Tsimring, Anomalous diffusion of particles in regular fields, *Phys. Lett. A* **151**:523 (1990).
9. H. Kook and J. D. Meiss, Diffusion in symplectic maps, *Phys. Rew. A* **41**:4143–4150 (1990).
10. J. R. Cary and J. D. Meiss, Rigorously diffusive deterministic maps, *Phys. Rev.* **24A**:2624–2668 (1981).
11. J. D. Meiss, J. D. Cary, C. Grebogi, J. D. Craford, A. N. Kaufman, and H. D. Abarbanel, Correlations of periodic area preserving maps, *Physica* **6D**:375 (1983).
12. N. I. Chernov, Ergonic and statistical properties of piecewise linear hyperbolic automorphism of the 2-torus, *J. Stat. Phys.* **69**:111 (1992).
13. S. Siboni, Rilassamento all'equilibrio in un sistema mixing e analisi di un modello di diffusione modulata, Ph.d. thesis, Università degli Studi di Bologna, Italy (1991).